Differential Calculus (Partial Differentiation)

A partial derivative of a function of several variables is ~~the~~ its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary).

Partial derivatives are used in vector calculus and differential geometry.

The partial derivative of a function $f(x, y, ---)$ with respect to the variable x is variously denoted by

$$f'_x,\ f_x,\ \partial_x f,\ D_x f,\ D_1 f,\ \frac{\partial}{\partial x} f,\ \text{or } \frac{\partial f}{\partial x}.$$

Sometimes, for $z = f(x, y, ---)$, the partial derivative of z with respect to x is denoted as $\frac{\partial z}{\partial x}$. Since a partial derivative generally has the same arguments as the original function, its functional dependence is sometimes explicitly signified by the notation, such as in:

$$f_x(x, y, ---),\ \frac{\partial f}{\partial x}(x, y, ---).$$

The symbol used to denote partial derivatives is ∂. One of the first known uses of this symbol in mathematics by Marquis de Condorcet from 1770, who used it for partial differences. The modern partial derivative notation was created by Adrien-Marie Legendre (1786) (although he later abandoned it, Carl Gustav Jacob Jacobi reintroduced the symbol in 1841.

Like ordinary derivatives, the partial derivative is defined as a limit. Let U be an open subset of R^n and $f: U \to R$ a function. The partial derivative of f at the point $a = (a_1, a_2 \ldots, a_n) \in U$ with respect to the i-th variable x_i is defined as

$$\frac{\partial}{\partial x_i} f(a) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

Even if all partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist at a given point a, the function need not be continuous there. However, if all partial derivatives exist in a neighborhood of a and are continuous there, then f is totally differentiable in that neighborhood and the total derivatives is continuous. In this case, it is said that f is a C^1 function. This can be used to generalize for vector valued functions, $f: U \to R^m$, by carefully using a componentwise argument.

The partial derivative $\frac{\partial f}{\partial x}$ can be seen as another function defined on U and can again be partially differentiated. If all mixed second order partial derivatives are continuous at a point (or on a set), f is termed C^2 function at that point (or on that set); In this case, the partial derivatives can be exchanged by Clairaut's theorem:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$